Solid state physics III

Exercise session 7

November 6th 2025

Landau levels in graphene: In this series we investigate the effect of a magnetic field on relativistic Dirac fermions, which emerge effectively in graphene at low energies.

References:

1. M. O. Goerbig, Electronic properties of graphene in a strong magnetic field, Rev. Mod. Phys. **83**, 1193 (2011)
2. M. O. Goerbig, The quantum Hall effect in graphene – a theoretical perspective, C. R. Physique **12** (2011)

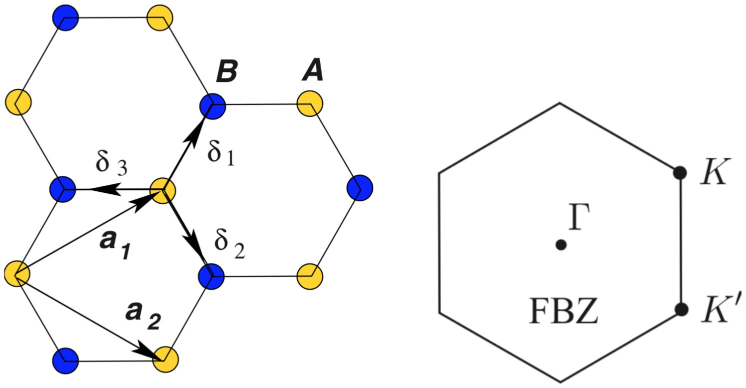

Figure 1: (Left) Honeycomb lattice. (Right) First Brillouin zone with the points $\Gamma = (0, 0)$, **K**, and **K'** indicated.

Graphene is a hexagonal lattice made of carbon atoms. Let us first derive the energy dispersion for the tight-binding model on the honeycomb lattice depicted in Fig. 1. The nearest-neighbour tight-binding model is

$$H = -t \sum_{\langle ij \rangle} \sum_{\sigma} \left[a^{\dagger}_{i\sigma} b_{j\sigma} + \text{H.c.} \right], \tag{1}$$

where $i \in A$ and $j \in B$ are nearest neighbours on the honeycomb lattice, and $a^{\dagger}_i$ and $b^{\dagger}_j$ (a_i and b_j) are the creation (annihilation) operators on the A and B sublattices, respectively. The spin index σ is dropped in the following. For an A-sublattice site, the three nearest-neighbour vectors are given by (see Fig. 1)

$$\boldsymbol{\delta}_1 = \frac{a}{2}(1, \sqrt{3}), \qquad \boldsymbol{\delta}_2 = \frac{a}{2}(1, -\sqrt{3}), \qquad \boldsymbol{\delta}_3 = a(-1, 0). \tag{2}$$

We define the Fourier-transformed operators

$$\begin{aligned} a^\dagger_{\mathbf{k}} &= \frac{1}{\sqrt{N}}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i} a^\dagger_i \qquad & b^\dagger_{\mathbf{k}} &= \frac{1}{\sqrt{N}}\sum_j e^{i\mathbf{k}\cdot\mathbf{r}_j} b^\dagger_j \\ a_{\mathbf{k}} &= \frac{1}{\sqrt{N}}\sum_i e^{-i\mathbf{k}\cdot\mathbf{r}_i} a_i \qquad & b_{\mathbf{k}} &= \frac{1}{\sqrt{N}}\sum_j e^{-i\mathbf{k}\cdot\mathbf{r}_j} b_j, \end{aligned} \tag{3}$$

where $\mathbf{r}_i$ is the position of site i.

1. Show that the Hamiltonian can be written as

$$H = \sum_{\mathbf{k}} \begin{pmatrix} a^\dagger_{\mathbf{k}} & b^\dagger_{\mathbf{k}} \end{pmatrix} H_{\mathbf{k}} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \end{pmatrix}, \qquad H_{\mathbf{k}} = \begin{pmatrix} 0 & \Delta_{\mathbf{k}}, \\ \Delta^*_{\mathbf{k}} & 0 \end{pmatrix} \tag{4}$$

   where $H_{\mathbf{k}}$ is the Bloch Hamiltonian, and calculate $\Delta_{\mathbf{k}}$.

2. Show that corresponding energy dispersion can be written as

$$\epsilon_{\mathbf{k},\pm} = \pm t\sqrt{3 + 2\cos\left(\sqrt{3}k_y a\right) + 4\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\cos\left(\frac{3}{2}k_x a\right)}. \tag{5}$$

3. The equation $\epsilon_{\mathbf{k},\pm} = 0$ is satisfied for the wave-vectors $\mathbf{k}$ at the six corners of the first Brillouin zone (FBZ). There are only two non-equivalent points (see Fig. 1):

$$\mathbf{K} = \frac{2\pi}{3a}(1, \frac{1}{\sqrt{3}}), \qquad \mathbf{K'} = \frac{2\pi}{3a}(1, -\frac{1}{\sqrt{3}}). \tag{6}$$

   Show that up to an overall (arbitrary) phase factor, we have

$$\Delta_{\mathbf{K}+\mathbf{k}} = \hbar v_F(-ik_x + k_y) + \mathcal{O}(k^2), \quad \Delta_{\mathbf{K'}+\mathbf{k}} = \hbar v_F(-ik_x - k_y) + \mathcal{O}(k^2), \tag{7}$$

   so that the Bloch Hamiltonian around **K** and **K'** can be written as

$$H^K = \hbar v_F(k_x\sigma_y + k_y\sigma_x), \quad H^{K'} = \hbar v_F(k_x\sigma_y - k_y\sigma_x), \tag{8}$$

   where $\hbar v_F = 3ta/2$, and σ_x and σ_y are Pauli matrices. Calculate the energy dispersion around the two points. At half-filling ($E_F = 0$), what is the density of states at the Fermi level?

4. In the non-relativistic case (see course), we introduced the operators $\hat{\eta}_x = \frac{l_B^2}{\hbar}\hat{\Pi}_y$ and $\hat{\eta}_y = -\frac{l_B^2}{\hbar}\hat{\Pi}_x$, where $l_B^2 = \hbar c/(eB)$. Then we defined

$$a = \frac{1}{\sqrt{2}l_B}(\hat{\eta}_x - i\hat{\eta}_y), \qquad a^\dagger = \frac{1}{\sqrt{2}l_B}(\hat{\eta}_x + i\hat{\eta}_y),$$

   or equivalently

$$\hat{\Pi}_x = i\frac{\hbar}{\sqrt{2}l_B}(a^\dagger - a), \qquad \hat{\Pi}_y = \frac{\hbar}{\sqrt{2}l_B}(a^\dagger + a). \tag{9}$$

   The Hamiltonian could then be written as $H = \hbar\omega_c(a^\dagger a + \frac{1}{2})$. The corresponding eigenstates $\{|n\rangle, n \in \mathbb{N}\}$ satisfy

$$a^\dagger a\,|n\rangle = n\,|n\rangle, \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad a\,|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{10}$$

   Show that for graphene in a perpendicular magnetic field, the Hamiltonians around **K** and **K'** read (using the substitution $\hbar\mathbf{k} \to \hat{\mathbf{\Pi}}$):

$$H^K = \hbar\omega_0 \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}, \qquad H^{K'} = -\hbar\omega_0 \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}. \tag{11}$$

5. Solve the eigenvalue equations

$$H^K \psi_n^K = \epsilon_n \psi_n^K, \quad \text{and} \quad H^{K'} \psi_n^{K'} = \epsilon_n \psi_n^{K'}, \quad \text{where} \quad \psi_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{12}$$

and u_n and v_n are linear combinations of $\{|m\rangle, m \in \mathbb{N}\}$. What are the energy levels ϵ_n? How do they differ from the energy levels of non-relativistic particles?

6. What is particular about ψ_0^K and $\psi_0^{K'}$? (How are they localized on the sublattices?) Here we defined the $n = 0$ states as the zero-energy states ($\epsilon_0 = 0$).